Computers and Mathematics with Applications 60 (2010) 1343–1350

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Classes of multivalent functions associated with a convolution operator

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ARTICLE INFO

Article history:
Received 24 November 2009
Received in revised form 7 June 2010
Accepted 7 June 2010

Keywords:
Hadamard product (or convolution)
Subordination
Linear operator
Convolution operator
Analytic functions
Starlike function
Convex function
Gamma function
Beta function
Gaussian hypergeometric function

ABSTRACT

We investigate several properties of the linear Aouf–Silverman–Srivastava operator and associated classes of multivalent analytic functions which were introduced and studied by Aouf et al. [M.K. Aouf, H. Silverman, H.M. Srivastava, Some families of linear operators associated with certain subclasses of multivalent functions, Comp. Math. Appl. 55 (2008) 535–549]. Several theorems are extensions of earlier results of the above paper.

1. Introduction

Let $\mathcal{A}(p)$, $p \in \mathbb{N}$, denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. We say that $f \in \mathcal{A}(p)$ is subordinate to $g \in \mathcal{A}(p)$, written $f \prec g$, if and only if there exists a Schwarz function ω, $\omega(0) = 0$ and $|\omega(z)| < 1$ in U such that $f(z) = g(\omega(z))$. Many classes of functions studied in geometric function theory can be described in terms of subordination. Let us define

$$\mathcal{S}_p^*(\varphi) := \left\{ f \in \mathcal{A}(p) : \frac{zf'(z)}{f(z)} \prec \varphi(z),\ z \in U \right\}, \tag{1}$$

$$\mathcal{K}_p(\varphi) := \left\{ f \in \mathcal{A}(p) : \left[1 + \frac{zf''(z)}{f'(z)} \right] \prec \varphi(z),\ z \in U \right\}, \tag{2}$$

where φ is analytic in U with $\varphi(0) = p$. For $\varphi(z) = \frac{1+z}{1-z}$ (1) and (2) become the well known classes $\mathcal{S}^*$, $\mathcal{K}$ of starlike and convex functions, respectively. For special choices for the functions φ we can obtain other classes investigated many times earlier. If we restrict our attention to the functions φ which map U onto a disc or a half-plane then we obtain the classes

$$\mathcal{S}_p^*\left(\frac{1+Az}{1+Bz}\right), \quad -1 \le B < A \le 1,$$

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0898-1221/$ – see front matter
doi:10.1016/j.camwa.2010.06.015

and

$$\mathcal{K}_p\left(\frac{1+Az}{1+Bz}\right), \quad -1 \le B < A \le 1,$$

introduced and investigated for $p = 1$ by Janowski [1,2]. These classes become the classes of starlike and convex functions of order α for $B = -1$ and $A = 1 - 2\alpha$ that were introduced by Robertson [3]. The paper [4] is dedicated to the case when $\varphi(U)$ is one of the conic regions, while in the paper [5] the set $\varphi(U)$ is connected to the lemniscate of Bernoulli. One can alter the conditions in (1) and (2) by setting Hf instead of f, where H is an operator on $\mathcal{A}(p)$, for example a convolution operator. For $f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p}$ and $g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}$ the Hadamard product (or convolution) is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+P}.$$

This product is associative, commutative and distributive over addition and $1/(1-z)$ is an identity for it.

Aouf, Silverman and Srivastava in [6] considered a linear convolution operator $L_p(a, c)$ introduced by Saitoh [7]:

$$L_p(a, c) : \mathcal{A}(p) \to \mathcal{A}(p)$$

through

$$L_p(a, c)f(z) = \varphi_p(a, c; z) * f(z),$$

where $a, c \in \mathbb{R}, c \neq 0, -1, -2, \ldots$ (one can consider complex a, c) and $\varphi_p(a, c; z)$ is defined by means of the hypergeometric function

$$\varphi_p(a, c; z) = z^p\, {}_2F_1(1, a, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p} \quad (z \in U),$$

where $(x)_n$ is the Pochhammer symbol

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = \begin{cases} 1 & \text{for} \quad n = 0, x \neq 0 \\ x(x+1)\cdots(x+n-1) & \text{for} \quad k \in \mathbb{N} = \{1, 2, 3, \ldots\}. \end{cases}$$

In [8] Carlson and Schaffer defined a linear operator $L(a, c) = L_1(a, c)$ with complex a, c. The Carlson–Schaffer operator contains the Ruscheweyh operator [9]

$$D^{\lambda} f(z) := \frac{z}{(1-z)^{\lambda+1}} * f(z) \quad (\lambda > -1;\ z \in U),$$

because $L(\lambda + 1; 1)f(z) = D^{\lambda} f(z)$. If $\lambda = n \in \{0, 1, 2, \ldots\}$, then the operator D^{λ} becomes

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!}$$

so $D^n f(z)$ is called the Ruscheweyh differential operator. Dziok and Srivastava in [10] considered a certain generalization of the operator $L(a, c) = L_1(a, c)$. Choi, Saigo and Srivastava in [11] defined by analogy with the Ruscheweyh operator the operator

$$I_{\lambda,\mu} : \mathcal{A}(p) \to \mathcal{A}(p)$$

through

$$I_{\lambda,\mu} f(z) = f_{\lambda,\mu}(z) * f(z) \quad (\lambda > -1, \mu > 0),$$

where

$$\frac{z}{(1-z)^{\lambda+1}} * f_{\lambda,\mu}(z) = \frac{z}{(1-z)^{\mu}}.$$

In particular, by taking $\lambda = n \in \{0, 1, 2, \ldots\}$ and $\mu = 2$ they obtained the operator considered earlier by Noor [12]. The Choi–Saigo–Srivastava operator $I_{\lambda,\mu}$ is a special case of the Carlson–Schaffer operator because $I_{\lambda,\mu} = L(\mu, \lambda+1)$. The authors of [11] obtained many interesting results and introduced the following classes of analytic functions for $\lambda > -1$, $\mu > 0$ and $\varphi, \psi \in \mathcal{N}$:

$$\mathcal{S}^*_{\lambda,\mu}(\varphi) := \left\{f : f \in \mathcal{A}(p) \text{ and } I_{\lambda,\mu} f(z) \in \mathcal{S}^*(\varphi)\right\},$$

$$\mathcal{K}_{\lambda,\mu}(\varphi) := \left\{f : f \in \mathcal{A}(p) \text{ and } I_{\lambda,\mu} f(z) \in \mathcal{K}(\varphi)\right\},$$

where

$$\mathcal{N} = \{\varphi : z\varphi \in \mathcal{A}(p),\ \mathfrak{Re}\varphi(z) > 0 \text{ for } z \in U \text{ and } \varphi \text{ is convex univalent}\}.$$

A function φ is called convex if $\varphi(U)$ is a convex set. Some applications involving these and other families of integral operators were also considered. In particular, the classes

$$\mathcal{S}^*_{\lambda,\mu}\left(\frac{1+Az}{1+Bz}\right) := \mathcal{S}^*_{\lambda,\mu}[A,B], \quad (-1 \le B < A \le 1)$$

and

$$\mathcal{K}_{\lambda,\mu}\left(\frac{1+Az}{1+Bz}\right) := \mathcal{K}_{\lambda,\mu}[A,B], \quad (-1 \le B < A \le 1)$$

were studied. Aouf, Silverman and Srivastava in [6] by means of the linear operator $L_p(a,c)$ defined the class $\mathcal{P}_{a,c}(A,B,\lambda,p)$ of functions $f \in \mathcal{A}(p)$ such that

$$\frac{1}{p-\lambda}\left[\frac{\left[L_p(a,c)f(z)\right]'}{z^{p-1}} - \lambda\right] \prec \frac{1+Az}{1+Bz} \quad (z \in U) \tag{3}$$

or, equivalently, where the following inequality holds true:

$$\left|\frac{\frac{\left[L_p(a,c)f(z)\right]'}{z^{p-1}} - p}{B\frac{\left[L_p(a,c)f(z)\right]'}{z^{p-1}} - [pB + (A-B)(p-\lambda)]}\right| < 1 \quad \text{for all } z \in U, \tag{4}$$

where $-1 \le B < A \le 1$ and $0 \le \lambda < p$.

In [6] the authors presented a long list of classes that are subclasses of $\mathcal{P}_{a,c}(A,B,\lambda,p)$ which where studied in many earlier works, so we refer the reader to the references in [6].

The aim of this paper is to give more results concerning the above class of multivalent functions. We continue and extend the considerations of the basic paper [6].

2. Main results

Theorem 1. *Let $a, c \in \mathbb{C}$, $c \ne 0, -1, -2, \ldots$, $-1 \le B < A \le 1$ and $0 \le \lambda < p$. A function $f \in \mathcal{A}(p)$ belongs to the class $\mathcal{P}_{a,c}(A,B,\lambda,p)$, defined in (3)–(4), if and only if*

$$\frac{(f * \widehat{\varphi})(z)}{z^p} + \frac{z}{p}\left[\frac{(f * \widehat{\varphi})(z)}{z^p}\right]' \prec \frac{1+Az}{1+Bz} \quad (z \in U), \tag{5}$$

where

$$\widehat{\varphi} = \widehat{\varphi}_p(a,c,\lambda)(z) = z^p + \frac{p}{p-\lambda}\sum_{n=1}^{\infty}\frac{(a)_n}{(c)_n}z^{n+p} \quad (z \in U). \tag{6}$$

Proof. Let $f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots$. From (3) we have

$$\begin{aligned}
\frac{1}{p-\lambda}\left[\frac{\left[L_p(a,c)f(z)\right]'}{z^{p-1}} - \lambda\right] &= \frac{1}{p-\lambda}\left[\frac{pz^{p-1} + \sum_{n=1}^{\infty}\frac{(a)_n}{(c)_n}(n+p)a_{n+p}z^{n+p-1}}{z^{p-1}} - \lambda\right] \\
&= \frac{f(z)}{z^p} * \left[1 + \sum_{n=1}^{\infty}\frac{n+p}{p-\lambda}\frac{(a)_n}{(c)_n}z^n\right] \\
&= \frac{f(z)}{z^p} * \left[\frac{\widehat{\varphi}_p(a,c,\lambda)(z)}{z^p} + \frac{z}{p}\left[\frac{\widehat{\varphi}_p(a,c,\lambda)(z)}{z^p}\right]'\right] \\
&= \frac{[f * \widehat{\varphi}_p(a,c,\lambda)](z)}{z^p} + \frac{z}{p}\left[\frac{[f * \widehat{\varphi}_p(a,c,\lambda)](z)}{z^p}\right]' \prec \frac{1+Az}{1+Bz} \quad (z \in U),
\end{aligned}$$

and therefore the left-hand sides of (3) and of (5) are the same. □

For $a = 0$ the function $\widehat{\varphi}$ becomes $\widehat{\varphi}(z) = z^p$ and the condition (5) is satisfied by each $f \in \mathcal{A}(p)$; thus $\mathcal{P}_{0,c}(A,B,\lambda,p) = \mathcal{A}(p)$. Therefore in the following considerations we assume that $a \ne 0$. Now we recall the following lemma which will be required in our next investigation.

Lemma 1. *Let h be an analytic and convex univalent function in U. Let f be analytic in U with $h(0) = f(0) = 1$. If*

$$f(z) + \frac{zf'(z)}{\gamma} \prec h(z) \quad (z \in U), \tag{7}$$

for $\gamma \neq 0$ *and* $\mathfrak{Re}[\gamma] \geq 0$, *then*

$$f(z) \prec g(z) = \frac{\gamma}{z^{\gamma}} \int_0^z t^{\gamma-1} h(t) \mathrm{d}t \prec h(z) \quad (z \in U). \tag{8}$$

Moreover, the function $g(z)$ *is convex univalent and it is the best dominant of* (8) *in the sense that if there exists a function* g_1 *such that* $f \prec g_1$, *then also* $g \prec g_1$.

The above lemma is due to Hallenbeck and Ruscheweyh [13].

Theorem 2. *Let* $a, c \in \mathbb{C} \setminus \{0\}, c \neq -1, -2, \ldots, 0 \leq \lambda < p$ *and* $\widehat{\varphi}$ *be given by* (6). *If a function* $f \in \mathcal{A}(p)$ *and a convex univalent function* h *satisfy*

$$\frac{(f * \widehat{\varphi})(z)}{z^p} + \frac{z}{p}\left[\frac{(f * \widehat{\varphi})(z)}{z^p}\right]' \prec h(z) \quad (z \in U), \tag{9}$$

then

$$\frac{(f * \widehat{\varphi})(z)}{z^p} \prec g(z) = \frac{p}{z^p} \int_0^z t^{p-1} h(t) \mathrm{d}t \prec h(z) \quad (z \in U). \tag{10}$$

Moreover, the function $g(z)$ *is convex univalent and it is the best dominant of* (10).

Proof. The subordination (10) is a simple consequence of Lemma 1. □

Corollary 1. *Let* $a, c \in \mathbb{C} \setminus \{0\}, c \neq -1, -2, \ldots, -1 \leq B < A \leq 1$ *and* $0 \leq \lambda < p$. *If a function* $f \in \mathcal{P}_{a,c}(A, B, \lambda, p)$ *and the function* $\widehat{\varphi}$ *is given by* (6), *then*

$$\frac{(f * \widehat{\varphi})(z)}{z^p} \prec g_p(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in U), \tag{11}$$

where

$$g_p(z) = 1 + \frac{p(A - B)}{p + 1} z + p(A - B) \sum_{n=2}^{\infty} \frac{(-B)^{n-1}}{p + n} z^n \quad (z \in U). \tag{12}$$

Moreover, the function $g_p(z)$ *is convex univalent and it is the best dominant of* (11).

Proof. We make use of Theorem 2. Substituting $h(t) = \frac{1+At}{1+Bt}$ in (10) and then integrating we can obtain (11). For $B \neq 0$ the function (12) becomes

$$g_p(z) = 1 + p(A - B) \sum_{n=1}^{\infty} \frac{(-B)^{n-1}}{p + n} z^n. \quad \square$$

If we consider the function f_p such that $(f_p * \widehat{\varphi})(z) = z^p g_p(z)$, then we obtain

$$f_p(z) = z^p + \frac{c(A - B)(p - \lambda)}{a(p + 1)} z^{p+1} + (A - B)(p - \lambda) \sum_{n=2}^{\infty} \frac{(c)_n (-B)^{n-1}}{(a)_n (p + n)} z^{n+p} \quad (z \in U), \tag{13}$$

whenever $a \neq 0, -1, -2, \ldots$. In this case $f_p \in \mathcal{P}_{a,c}(A, B, \lambda, p)$. The principle of subordination says that $f \prec g$ with univalent g is equivalent to $f(|z| < r) \subset g(|z| < r), f(0) = g(0)$, for all $r \in (0, 1)$. Because the function g_p is convex univalent with real coefficients, we have that $g_p(|z| < r)$ is a convex set symmetric with respect to the real axis with

$$g_p(-r) < \mathfrak{Re}[g_p(z)] < g_p(r) \quad (|z| < r < 1),$$

and hence we have the following corollary.

Corollary 2. *If a function* f *belongs to the class* $\mathcal{P}_{a,c}(A, B, \lambda, p)$, *then*

$$\begin{aligned} 1 - \frac{p(A - B)}{p + 1} r + p(A - B) \sum_{n=2}^{\infty} \frac{(-B)^{n-1}}{p + n} (-r)^n &= g_p(-r) < \mathfrak{Re}\left[\frac{(f * \widehat{\varphi})(z)}{z^p}\right] < g_p(r) \\ &= 1 + \frac{p(A - B)}{p + 1} r + p(A - B) \sum_{n=2}^{\infty} \frac{(-B)^{n-1}}{p + n} r^n \quad (|z| < r < 1). \end{aligned} \tag{14}$$

For finding the sums of the series in (14) notice that for $-1 < x \leq 1$ we have

$$\sum_{n=0}^{\infty} \frac{(-x)^n}{n + p} = \frac{1}{x^p}\left[(-1)^{p-1} \log(1 + x) + \sum_{n=1}^{p-1} \frac{(-1)^{n-1} x^{p-n}}{p - n}\right].$$

Thus the left-hand side of (14) we can write for $rB \neq 0$ in the form

$$1-\frac{p\,(A-B)}{p+1}r+p\,(A-B)\sum_{n=2}^{\infty}\frac{(-B)^{n-1}}{p+n}(-r)^{n}=1+p\left(1-\frac{A}{B}\right)\left[\frac{(-1)^{p-1}\log(1+rB)}{(rB)^{p}}+\sum_{n=0}^{p-1}\frac{(-1)^{n-1}}{(p-n)(rB)^{n}}\right].$$

Moreover, we have

$$\sum_{n=0}^{\infty}\frac{x^{n}}{n+p}=\frac{1}{x^{p}}\left[-\log(1-x)-\sum_{n=1}^{p-1}\frac{x^{n}}{n}\right]\quad(-1\leq x<1),$$

so the right-hand side of (14), for $rB \neq 0$, can be reformulated into the form

$$1+\frac{p\,(A-B)}{p+1}r+p\,(A-B)\sum_{n=2}^{\infty}\frac{(-B)^{n-1}}{p+n}r^{n}=\frac{A}{B}+p\left(\frac{A}{B}-1\right)\left[\frac{\log(1-rB)}{(rB)^{p}}+\sum_{n=1}^{p-1}\frac{(rB)^{n-p}}{n}\right].$$

Now we recall some known results which will be required in Lemma 2. We start with the usual properties of the hypergeometric function. It is known that the Gaussian hypergeometric function

$${}_2F_1(\alpha,\beta,\gamma;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}z^{n}\quad(z\in U),$$

has for $\mathfrak{Re}[\gamma]>\mathfrak{Re}[\alpha]>0$ an integral representation (see for example [14], Chap. XIV) of the form

$${}_2F_1(\alpha,\beta,\gamma;z)=\int_0^1(1-tz)^{-\beta}\mathrm{d}\mu(t),\tag{15}$$

where

$$\mathrm{d}\mu(t)=\frac{\Gamma(\gamma)t^{\alpha-1}(1-t)^{\gamma-\alpha-1}}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\mathrm{d}t$$

satisfies

$$\int_0^1\mathrm{d}\mu(t)=\frac{\Gamma(\gamma)\mathcal{B}(\alpha,\gamma-\alpha)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}=1,$$

where $\mathcal{B}$ is the Beta function and Γ is the Gamma function. If $|w|<1$ then $\mathfrak{Re}[1/(1-w)]>1/2$, and therefore for $\beta=1$, $\gamma>\alpha>0$ and $|z|<1$ (15) gives

$$\begin{aligned}\mathfrak{Re}\left[\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n}z^{n}\right]&=\mathfrak{Re}\left[{}_2F_1(\alpha,1,\gamma;z)\right]\\&=\mathfrak{Re}\left[\int_0^1\frac{1}{(1-tz)}\mathrm{d}\mu(t)\right]\\&=\int_0^1\mathfrak{Re}\frac{1}{(1-tz)}\mathrm{d}\mu(t)>\int_0^1\frac{1}{2}\mathrm{d}\mu(t)=\frac{1}{2}.\end{aligned}\tag{16}$$

It is clear that (16) is also satisfied for $\alpha=\gamma$. Moreover, after some adaptations, Theorem 4.5(f) [15] says that the function

$$\varphi(\alpha,\gamma;z)=\sum_{n=0}^{\infty}\frac{(\alpha-1)_n}{(\gamma-1)_n}z^{n}$$

is convex whenever $0\neq\alpha$, $-1<\alpha<1$ and $\gamma>3+|\alpha|$, so in this case

$$\frac{\gamma-1}{\alpha-1}\left[\varphi(\alpha,\gamma;z)-1\right]=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n}z^{n+1}$$

belongs to the class $\mathcal{K}$ of convex univalent functions. It is known that $f\in\mathcal{K}$ follows $\mathfrak{Re}[f(z)/z]>1/2$ for $z\in U$; thus

$$\mathfrak{Re}\left[\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\gamma)_n}z^{n}\right]>\frac{1}{2}\quad\text{for all } z\in U.\tag{17}$$

The problem of finding complex α,γ satisfying (17) was partially solved in [9] by considerations of prestarlike functions. It is known that if f is a prestarlike function, then $\mathfrak{Re}[f(z)/z]>1/2$. Reformulating Theorem 2.12 in [9] we obtain that if $\mathfrak{Im}\alpha=\mathfrak{Im}\gamma$, and $\mathfrak{Re}\gamma\geq\max\{\mathfrak{Re}\alpha,1-\mathfrak{Re}\alpha\}$, then (17) is satisfied. It is clear that (17) is also satisfied for complex $\alpha=\gamma$, excluding $0,-1,-2,\ldots$. In this way we have then proved the next lemma.

Lemma 2. *If one of the following conditions is satisfied:*
(1) $0 < \alpha < \gamma$,
(2) $0 \neq \alpha$, $-1 < \alpha < 1$ *and* $\gamma > 3 + |\alpha|$,
(3) $\Im\mathfrak{m}\alpha = \Im\mathfrak{m}\gamma$ *and* $\Re\mathfrak{e}\gamma \geq \max\{\Re\mathfrak{e}\alpha, 1 - \Re\mathfrak{e}\alpha\}$,
(4) $\alpha = \gamma \in \mathbb{C} \setminus \{\dots, -2, -1, 0\}$,
then (17) *holds true.*

Corollary 3. *Let* $a, c \in \mathbb{C} \setminus \{0\}$, $a, c \neq -1, -2, \dots, 0 \leq \lambda < p$ *and* $\widehat{\varphi}$ *be given by* (6). *Then there exists*

$$\left[\frac{\widehat{\varphi}(z)}{z^p}\right]^{[-1]} \quad \text{such that} \quad \left[\frac{\widehat{\varphi}(z)}{z^p}\right]^{[-1]} * \frac{\widehat{\varphi}(z)}{z^p} = \frac{1}{1-z}.$$

Moreover, if additionally one of the following conditions is satisfied:
(1) $a, c \in \mathbb{R}$ *and* $0 < c < a$,
(2) $a, c \in \mathbb{R}$ *and* $0 \neq |c| < 1$ *and* $a > 3 + |c|$,
(3) $\Im\mathfrak{m}c = \Im\mathfrak{m}a$, *and* $\Re\mathfrak{e}a \geq \max\{\Re\mathfrak{e}c, 1 - \Re\mathfrak{e}c\}$,
(4) $a = c \in \mathbb{C} \setminus \{\dots, -2, -1, 0\}$,
then

$$\Re\mathfrak{e}\left[\frac{\widehat{\varphi}(z)}{z^p}\right]^{[-1]} = \Re\mathfrak{e}\left[1 + \frac{p-\lambda}{p}\sum_{n=1}^{\infty}\frac{(c)_n}{(a)_n}z^n\right] > \frac{1}{2} \quad \text{for all } z \in U. \tag{18}$$

Proof. It is clear that our assumptions are sufficient for the existence and convergence of the series $[\widehat{\varphi}(z)/z^p]^{[-1]}$ in $|z| < 1$. If we write

$$\left[\frac{\widehat{\varphi}(z)}{z^p}\right]^{[-1]} = 1 - \frac{(p-\lambda)}{p} + \frac{(p-\lambda)}{p}\sum_{n=0}^{\infty}\frac{(c)_n}{(a)_n}z^n,$$

then we can see that Lemma 2 follows (18). □

The following lemma was obtained by Singh and Singh [16] in 1989 and is a generalization of an earlier result of Wilf [17] on subordinating factor sequences for convex maps of the unit circle.

Lemma 3. *If a function* p, *with* $p(0) = 1$, *is analytic and* $\Re\mathfrak{e}p(z) > 1/2$ *in* U, *then for an analytic function* F *we have* $(p * F)(U) \subset coF(U)$, *where* coX *denotes the convex hull of the set* X. *If* F, *with* $F(0) = 1$, *is a convex univalent function, then this means that if* $f \prec F$, *then* $p * f \prec F$.

Theorem 3. *Under the assumptions of* Corollary 3, *if* $f \in \mathcal{P}_{a,c}(A, B, \lambda, p)$, *then*

$$\frac{f(z)}{z^p} \prec g_p(z) \prec \frac{1+Az}{1+Bz} \quad (z \in U), \tag{19}$$

where $g_p(z)$ *is given in* (12).

Proof. We have the subordination (11), with convex univalent function g_p (12), so by Lemma 3 the subordination (11) is preserved under convolution of its left-hand side with the function satisfying $\Re\mathfrak{e}\,[\widehat{\varphi}/z^p]^{[-1]} > \frac{1}{2}$. After a simple calculation, using Corollary 3, we can obtain (19). □

A few of the inclusion properties connecting classes $\mathcal{P}_{a,c}(A, B, \lambda, p)$ with different parameters were proved in [6] using the Jack's Lemma [18]. Here we are going to obtain a result stronger than that in [6] using another method of proof.

Theorem 4. *Let* $a, c \in \mathbb{C} \setminus \{0\}$, $c \neq -1, -2, \dots, -1 \leq B < A \leq 1$ *and* $0 \leq \lambda < p$. *Then we have*

$$\frac{(f * \widehat{\varphi}(a+1))(z)}{z^p} + \frac{z}{p}\left[\frac{(f * \widehat{\varphi}(a+1))(z)}{z^p}\right]' \prec \frac{1+Az}{1+Bz} \quad (z \in U), \tag{20}$$

if and only if for $z \in U$

$$\left\{\frac{(f * \widehat{\varphi}(a))(z)}{z^p} + \frac{z}{p}\left[\frac{(f * \widehat{\varphi}(a))(z)}{z^p}\right]'\right\} + \frac{z}{a}\left\{\frac{(f * \widehat{\varphi}(a))(z)}{z^p} + \frac{z}{p}\left[\frac{(f * \widehat{\varphi}(a))(z)}{z^p}\right]'\right\}' \prec \frac{1+Az}{1+Bz}, \tag{21}$$

where we define for simplicity $\widehat{\varphi}(a+1) = \widehat{\varphi}_p(a+1, c, \lambda)(z)$, $\widehat{\varphi}(a) = \widehat{\varphi}_p(a, c, \lambda)(z)$.

Proof. We want to show the equality of the left-hand sides of (20) and (21). Notice that

$$\widehat{\varphi}(a+1) = \widehat{\varphi}(a) * \left[z^p + \sum_{k=1}^{\infty}\frac{a+k}{a}z^{k+p}\right].$$

Using this and some algebraic properties of the convolution we obtain

$$\begin{aligned}
&\frac{(f*\widehat{\varphi}(a+1))\,(z)}{z^p}+\frac{z}{p}\left[\frac{(f*\widehat{\varphi}(a+1))\,(z)}{z^p}\right]'\\
&=\frac{(f*\widehat{\varphi}(a))\,(z)}{z^p}*\left[1+\sum_{k=1}^{\infty}\frac{a+k}{a}z^k\right]+\frac{1}{p}\frac{(f*\widehat{\varphi}(a))\,(z)}{z^p}*\left[\sum_{k=1}^{\infty}\frac{(a+k)k}{a}z^k\right]\\
&=\frac{(f*\widehat{\varphi}(a))\,(z)}{z^p}*\left[1+\sum_{k=1}^{\infty}\left(1+\frac{k}{p}+\frac{k}{a}+\frac{k^2}{ap}\right)z^k\right]\\
&=\frac{(f*\widehat{\varphi}(a))\,(z)}{z^p}*\left[1+\sum_{k=1}^{\infty}z^k+\frac{1}{p}\sum_{k=1}^{\infty}kz^k\right]+\frac{1}{a}\frac{(f*\widehat{\varphi}(a))\,(z)}{z^p}*\left[\sum_{k=1}^{\infty}kz^k+\frac{1}{p}\sum_{k=1}^{\infty}k^2z^k\right]\\
&=\left\{\frac{(f*\widehat{\varphi}(a))\,(z)}{z^p}+\frac{z}{p}\left[\frac{(f*\widehat{\varphi}(a))\,(z)}{z^p}\right]'\right\}+\frac{z}{a}\left\{\frac{(f*\widehat{\varphi}(a))\,(z)}{z^p}+\frac{z}{p}\left[\frac{(f*\widehat{\varphi}(a))\,(z)}{z^p}\right]'\right\}'. \qquad\square
\end{aligned}$$

Corollary 4. *Let $a, c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$, $-1 \leq B < A \leq 1$ and $0 \leq \lambda < p$. If $a = 0$ or $\mathfrak{Re}[a] > 0$, then*

$$\mathcal{P}_{a+1,c}(A, B, \lambda, p) \subset \mathcal{P}_{a,c}(A, B, \lambda, p). \tag{22}$$

Moreover if $f \in \mathcal{P}_{a+1,c}(A, B, \lambda, p)$ and $a \neq 0$, then

$$\frac{(f*\widehat{\varphi}(a))\,(z)}{z^p}+\frac{z}{p}\left[\frac{(f*\widehat{\varphi}(a))\,(z)}{z^p}\right]' \prec g_a(z) \prec \frac{1+Az}{1+Bz} \quad (z \in U), \tag{23}$$

where

$$g_a(z)=\frac{a}{z^a}\int_0^z t^{a-1}\frac{1+At}{1+Bt}\mathrm{d}t = 1+\frac{a\,(A-B)}{a+1}z+a\,(A-B)\sum_{n=2}^{\infty}\frac{(-B)^{n-1}}{a+n}z^n. \tag{24}$$

Moreover, the function (24) is convex univalent and it is the best dominant of (23).

Proof. The inclusion (22) is trivial for $a = 0$ because in this case $\mathcal{P}_{a,c}(A, B, \lambda, p) = \mathcal{A}(p)$. Let $f \in \mathcal{P}_{a+1,c}(A, B, \lambda, p)$ with $a \neq 0$. Then by Theorem 1, f satisfies (20); thus by Theorem 4, f satisfies (21). Using Lemma 1 with $\gamma = a$, $h(z) = \frac{1+Az}{1+Bz}$ in (21) we directly obtain the subordination (23) and its best dominant (24). By Theorem 1 the subordination (23) gives immediately that $f \in \mathcal{P}_{a,c}(A, B, \lambda, p)$, so we finally obtain (22). $\square$

The above result improves another one from [6], where the authors proved (22) for $a > 0$. If we consider the behaviour of the parameter c, then we see that

$$\widehat{\varphi}_p(a, c, \lambda)(z) = \widehat{\varphi}_p(a, c+1, \lambda)(z) * \left[z^p+\sum_{k=1}^{\infty}\frac{c+k}{c}z^{k+p}\right].$$

In the same manner as Corollary 4, we can obtain its analogous form for the parameter c.

Corollary 5. *Let $a, c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$, $-1 \leq B < A \leq 1$ and $0 \leq \lambda < p$. If $\mathfrak{Re}[c+1] > 0$, then*

$$\mathcal{P}_{a,c}(A, B, \lambda, p) \subset \mathcal{P}_{a,c+1}(A, B, \lambda, p). \tag{25}$$

Moreover if $f \in \mathcal{P}_{a,c}(A, B, \lambda, p)$, then

$$\frac{(f*\widehat{\varphi}(c+1))\,(z)}{z^p}+\frac{z}{p}\left[\frac{(f*\widehat{\varphi}(c+1))\,(z)}{z^p}\right]' \prec g_{c+1}(z) \prec \frac{1+Az}{1+Bz} \quad (z \in U), \tag{26}$$

where

$$\begin{aligned}
g_{c+1}(z) &= \frac{c+1}{z^{c+1}}\int_0^z t^{c}\frac{1+At}{1+Bt}\mathrm{d}t\\
&= 1+\frac{(c+1)\,(A-B)}{c+2}z+(c+1)\,(A-B)\sum_{n=2}^{\infty}\frac{(-B)^{n-1}}{c+1+n}z^n.
\end{aligned} \tag{27}$$

Moreover, the function (27) is convex univalent and it is the best dominant of (26).

We have considered the inclusion relations with distance of parameters equal 1. Finding sufficient conditions on a, b such that $\mathcal{P}_{a,c}(A, B, \lambda, p) \subset \mathcal{P}_{b,c}(A, B, \lambda, p)$ in general seams to be difficult. If we restrict our considerations to certain cases of the parameters a, b, then we can obtain partial results.

Theorem 5. *Let $c \in \mathbb{C} \setminus \{0, -1, -2, \ldots, \}$. If one of the following conditions is satisfied:*

(1) $0 < b < a$,
(2) $0 \neq b$, $-1 < b < 1$ *and* $a > 3 + |b|$,
(3) $\mathfrak{Im}\, b = \mathfrak{Im}\, a$ *and* $\mathfrak{Re}\, a \geq \max\{\mathfrak{Re}\, b, 1 - \mathfrak{Re}\, b\}$,
(4) $b = a \in \mathbb{C} \setminus \{, \ldots, -2, -1, 0\}$,

then we have

$$\mathcal{P}_{a,c}(A, B, \lambda, p) \subset \mathcal{P}_{b,c}(A, B, \lambda, p). \tag{28}$$

Proof. For simplicity let us define $\widehat{\varphi}_p(a, c, \lambda)(z) = \widehat{\varphi}(a)$, $\widehat{\varphi}_p(b, c, \lambda) = \widehat{\varphi}(b)(z)$. Under our assumptions from Lemma 2 we have

$$\mathfrak{Re}\left[\sum_{k=0}^{\infty} \frac{(b)_k}{(a)_k} z^k\right] = \mathfrak{Re}\left[\frac{\widehat{\varphi}(b)}{z^p}\right] * \left[\frac{\widehat{\varphi}(a)}{z^p}\right]^{[-1]} > \frac{1}{2} \quad \text{for all } z \in U. \tag{29}$$

If $f \in \mathcal{P}_{a,c}(A, B, \lambda, p)$, then

$$\frac{(f * \widehat{\varphi}(a))(z)}{z^p} + \frac{z}{p}\left[\frac{(f * \widehat{\varphi}(a))(z)}{z^p}\right]' \prec \frac{1 + Az}{1 + Bz} \quad (z \in U) \tag{30}$$

and $F(z) = (1 + Az)/(1 + Bz)$ is convex univalent. Making use of (29) and (30) and Lemma 3 we obtain

$$\begin{aligned} \frac{(f * \widehat{\varphi}(b))(z)}{z^p} + \frac{z}{p}\left[\frac{(f * \widehat{\varphi}(b))(z)}{z^p}\right]' &= \left\{\frac{(f * \widehat{\varphi}(a))(z)}{z^p} + \frac{z}{p}\left[\frac{(f * \widehat{\varphi}(a))(z)}{z^p}\right]'\right\} * \left\{\sum_{k=0}^{\infty} \frac{(b)_k}{(a)_k} z^k\right\} \\ &\prec \frac{1 + Az}{1 + Bz} \quad (z \in U), \end{aligned}$$

and thus $f \in \mathcal{P}_{b,c}(A, B, \lambda, p)$. $\square$

Acknowledgements

I would like to express sincere thanks to the referees for careful reading and suggestions which helped me to improve the paper.

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